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NUMBER THEORY.

When this issue was made up no solutions had been received for numbers 189, 191, 192, 196, 201, and 202. Please give attention to these.

208. Proposed by E. T. BELL, Seattle, Washington.

If an odd number is perfect, it cannot be the sum of two squares.

209. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that the difference of the sixth powers of two integers cannot be the square of an integer

210. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

If a and b are relatively prime and $(a + b)$ is even, then $(a - b)ab(a + b) \equiv 0 \pmod{24}$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

394. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Solve the equation $\sin^2 x \sin^2 2x = 5/16$.

SOLUTION BY R. M. MATHEWS, Riverside, California.

Taking the square root of each side of the given equation we obtain

$$(1) \sin x \sin 2x = \frac{\sqrt{5}}{4} \quad \text{and} \quad (2) \sin x \sin 2x = -\frac{\sqrt{5}}{4}.$$

From (1),

$$2 \sin^2 x \cos x = \frac{\sqrt{5}}{4}, \quad \text{or} \quad (3) \cos^3 x - \cos x + \frac{\sqrt{5}}{8} = 0.$$

Inspection shows that $-\frac{\sqrt{5}}{2}$ is a value of $\cos x$ in (3). Eliminating this impossible solution, we have the quadratic

$$(4) \cos^2 x - \frac{\sqrt{5}}{2} \cos x + \frac{1}{4} = 0,$$

the roots of which are $\cos x = \frac{\sqrt{5} + 1}{4}$ and $\cos x = \frac{\sqrt{5} - 1}{4}$.

The solutions of (2) are the negatives of those of (1) just found.

The side of a regular inscribed decagon is $(\sqrt{5} - 1)/2$, whence $(\sqrt{5} - 1)/4 = \cos 72^\circ$. From this, it is easily shown that $(\sqrt{5} + 1)/4 = \cos 36^\circ$.

Since in the original equations the functions are all squared, the signs which they may have in particular quadrants are immaterial. Hence

$$x = n\pi \pm \frac{1}{5}\pi, \quad x = n\pi \pm \frac{2}{5}\pi,$$

where n is any positive or negative integer, are the real values satisfying the equation.

Also solved by C. E. GITHENS, HORACE OLSON, ALBERT R. NAUER, RICHARD MORRIS, A. M. HARDING, ELMER SCHUYLER, S. W. REAVES, W. C. EELLS, LEROY M. COFFIN, A. H. HOLMES, F. M. MORGAN, H. C. FEEMSTER, J. L. RILEY, H. E. TREFETHEN, and the PROPOSER.

395. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the system of equations

$$x_1^2 x_2 = a_1, \quad x_2^2 x_3 = a_2, \quad x_3^2 x_4 = a_3, \quad \dots, \quad x_n^2 x_1 = a_n.$$

I. SOLUTION BY THOMAS E. MASON, Indiana University.

Assuming each a different from zero and expressing each x in terms of x_1 from the equations in order, we have

$$x_2 = \frac{a_1}{x_1^2}, \quad x_3 = \frac{a_2 x_1^4}{a_1^2}, \quad x_4 = \frac{a_3 a_1^4}{a_2^2 x_1^8}, \quad \dots, \quad x_i = \frac{a_{i-1} a_{i-3}^4 a_{i-5}^{16} \dots}{a_{i-2}^2 a_{i-4}^8 a_{i-6}^{32} \dots} x_1^{(-2)^{i-1}}.$$

Continuing this substitution we would obtain

$$x_{n+1} = \frac{a_n a_{n-2}^4 a_{n-4}^{16} \dots}{a_{n-1}^2 a_{n-3}^8 a_{n-5}^{32} \dots} x_1^{(-2)^n}.$$

But from the last of the given equations, we see that x_{n+1} is defined to be x_1 . Hence we have

$$x_1 = \frac{a_n a_{n-2}^4 a_{n-4}^{16} \dots}{a_{n-1}^2 a_{n-3}^8 a_{n-5}^{32} \dots} x_1^{(-2)^n}, \quad \text{or} \quad x_1^{2^n + (-1)^{n-1}} = \left(\frac{a_n a_{n-2}^4 a_{n-4}^{16} \dots}{a_{n-1}^2 a_{n-3}^8 a_{n-5}^{32} \dots} \right)^{(-1)^{n-1}}.$$

This can be solved by the ordinary trigonometric method and the $2^n + (-1)^{n-1}$ roots found. For each value of x_1 there will be corresponding values of x_i found by substituting the value of x_1 in the formula above for x_i .

Solved in like manner by H. C. FEEMSTER, A. H. HOLMES, and S. W. REAVES.

II. SOLUTION BY F. C. REISLER and I. A. BARNETT, University of Chicago.

Given the system of equations:

$$x_1^2 x_2 = a_1, \quad x_2^2 x_3 = a_2, \quad \dots, \quad x_n^2 x_1 = a_n.$$

If no $a = 0$, the following law is found to be true for all cases through $n = 4$.

$$x_k^{2^{n-(-1)^n}} = \prod_{l=0}^{n-1} a_{k+l}^{2^{n-1-l}(-1)^l}, \quad (1)$$

where $k = 1 \dots n$, and a_{n+q} is defined as a_q .

In order to show that (1) holds true for all values of n , we assume that it holds for $n = m$, that is,

$$x_k^{2^{m-(-1)^m}} = \prod_{l=0}^{m-1} a_{k+l}^{2^{m-1-l}(-1)^l}, \quad (2)$$

and prove that it holds for $n = m + 1$.

To do this we take a new set of $m + 1$ equations

$$y_1^2 y_2 = a_1, \quad y_2^2 y_3 = a_2, \quad \dots, \quad y_m^2 y_{m+1} = a_m, \quad y_{m+1}^2 y_1 = a_{m+1}.$$

From these and the given equations we obtain the new system,

$$x_1^2 x_2 = y_1^2 y_2, \quad x_2^2 x_3 = y_2^2 y_3, \quad \dots, \quad x_m^2 x_1 = y_m^2 y_{m+1}, \quad a_{m+1} = y_{m+1}^2 y_1, \quad (3)$$

where y_{m+1+p} is defined as y_p .

The solution of this system of equations is given by (2) as follows:

$$x_k^{2^{m-(-1)^m}} = \prod_{l=0}^{m-1} [y_{k+l}^2 y_{k+l+1}]^{2^{m-1-l}(-1)^l}.$$

In this product all of the factors except the first in the first couplet and the last in the last couplet become unity when combined in pairs, as can be seen by examining any such pair, for example,

$$(y_{k+l+1})^{2^{m-l-1}(-1)^l} (y_{k+l+1}^2)^{2^{m-l-2}(-1)^{l+1}},$$

in which the second factor may be written $(y_{k+l+1})^{2^{m-l-1}(-1)^{l+1}}$, and hence the sum of the exponents is zero no matter whether m is odd or even.

Therefore

$$x_k^{2^m(-1)^m} = y_k^{2^m} y_{k+m}^{(-1)^{m-1}} = \prod_{l=0}^{m-1} a_{k+l}^{(-1)^{l_2} m-1-l}. \quad (4)$$

Solve (4) for y_k^{2m} , square both sides, put $y_{m+k}^2 = (a_{m+k})/y_k$ as given by the last of equations (3), and we obtain, after a little reduction,

$$y_k^{2^{m+1}+(-1)^m} = \prod_{l=0}^m a_{k+l}^{(-1)^l 2^{m-l}}. \quad (5)$$

We see that (5) is the same as (2) with m replaced by $m+1$. Hence, if (1) holds for $n = m$, it also holds for $n = m+1$. But it has been verified that (1) holds for $n = 1, 2, 3, 4$. Hence it holds for all positive integral values of n .

In addition to the two interesting solutions of this problem given above, still another solution is worthy of reproducing, though the author is unknown, having neglected to *attach his name to his work*. If he sent a letter, it became separated from the solution. Will contributors *please follow the printed form* in sending in solutions. Also see full suggestions in October, 1913, issue.

III. Taking the logarithm of each member of each equation, we have the following set of n simultaneous equations in n unknowns.

$$\begin{array}{l} 2 \log x_1 + \log x_2 \dots\dots\dots = \log a_1 \\ \dots\dots\dots 2 \log x_2 + \log x_3 \dots\dots\dots = \log a_2 \\ \dots\dots\dots 2 \log x_3 + \log x_4 \dots\dots\dots = \log a_3 \\ \dots\dots\dots \\ \dots\dots\dots \\ \log x_1 \dots\dots\dots + 2 \log x_n = \log a_n. \end{array}$$

Then, by principles of determinants, we have

$$\log x_1 \begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 2 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 2 \end{vmatrix} = \begin{vmatrix} \log a_1 & 1 & 0 & 0 & \cdots & 0 \\ \log a_2 & 2 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & & & & & \\ \log a_n & 0 & \cdots & \cdots & 2 \end{vmatrix}$$

$$\log x_j \begin{vmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 & \cdots & \log a_1 & 0 & \cdots & 0 \\ 0 & 2 & 1 & \cdots & \log a_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \log a_j & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \log a_n & 0 & \cdots & 2 \end{vmatrix}.$$

Denoting the determinant in the left member of the last equation by D and the one in the right member by A_j , we have

$$\log x_j = \frac{A_j}{D},$$

or

$$x_j = e^{\frac{A_j}{D}} \quad (j = 1, 2, \cdots, n).$$

Also solved by the PROPOSER, who used logarithms as in the last solution, though he did not use determinants.

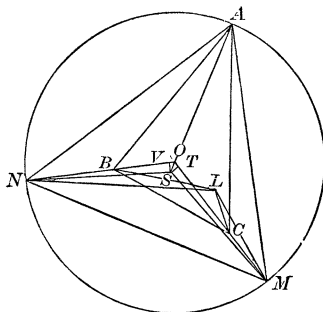
GEOMETRY.

424. Proposed by H. E. TREFETHEN, Colby College.

In a given triangle ABC , determine by geometric demonstration the point O such that the sum of the distances, $AO + BO + CO$, shall be a minimum.

SOLUTION BY M. E. GRABER, Heidelberg University.

On each of the three sides of the triangle ABC , describe a segment of a circle to contain an angle of 120° . The arcs of these circles intersect at a point O , about which the angles are each 120° . Then $AO + BO + CO$ is a minimum. Suppose $OA > OB$ or OC . Using OA as a radius describe a circumference. Produce OC and OB to meet the circumference at M and N . Let L be any other point in ABC .



Draw LC , LM , LB , LN , AM , MN , NA . Then $AM = MN = NA$ and $LA + LN + LM > OA + ON + OM$. In order to prove this relation, it is evidently sufficient to prove that $OA + ON + OM <$ the sum of the distances from S (the vertex of an isosceles Δ on MN with the same altitude as ΔLMN)